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Theorem 1. *With the Axiom of Choice, there exists an undetermined game on* $\omega^{<\omega}$.

Proof. Notice that a strategy is a function $\sigma : \omega^{<\omega} \to \omega^{<\omega}$. Thus, the set

$$S := \{\sigma : \sigma \text{ is a strategy }\} = \left(\omega^{<\omega}\right)^{\omega^{<\omega}},$$

and hence $|S| = c$ (the cardinality of the continuum). With AC we well-order $S = \{\sigma_\alpha : \alpha < \mathbf{C}\}$ (here, $\mathbf{C}$ is the first ordinal with cardinality c).

Now, we construct the payoff set A for the undetermined game. Let $\beta < \mathbf{C}$ be an ordinal, and suppose for all $\alpha < \beta$, $A_{\alpha+1}$,$B_{\alpha+1} \subset \omega^\omega$, have been constructed so that

- there exists $x_\alpha \in A_{\alpha+1}$ such that x_α respects σ_α
- there exists $y_\alpha \in B_{\alpha+1}$ such that y_α respects σ_α
- $A_{\alpha+1} \cap B_{\alpha+1} = \emptyset$
- For all $\gamma < \alpha$, $A_{\gamma+1} \subset A_{\alpha+1}$, and $B_{\gamma+1} \subset B_{\alpha+1}$
- $|A_{\alpha+1}| = |B_{\alpha+1}| = |\alpha + 1|$

Now, consider the strategy σ_β. It is clear that there are c many ways for one player to play against σ_β (the strategy σ_β is only employed on his opponent's move). If β is a limit ordinal, define

$$A_\beta := \bigcup_{\alpha<\beta} A_{\alpha+1},$$

$$B_\beta := \bigcup_{\alpha<\beta} B_{\alpha+1}.$$

Otherwise A_β, B_β are already defined.

Since

$$|A_\beta \cup B_\beta| < c,$$

there exists $x_\beta \in \omega^\omega$ that respects σ_β with $x \notin A_\beta \cup B_\beta$. Define

$$A_{\beta+1} := A_\beta \cup \{x_\beta\}.$$

Similarly find $y_\beta \notin A_\beta \cup B_\beta$, $y_\beta \neq x_\beta$, y_β respecting σ_β. Define

$$B_{\beta+1} := B_\beta \cup \{y_\beta\}.$$

Then $|A_{\beta+1}| = |\beta + 1| = |B_{\beta+1}|$, and the other properties also hold.
Finally, let

$$A := \bigcup_{\alpha < \mathbf{C}} A_{\alpha+1}.$$

With A as the payoff set, neither player can have a winning strategy, since for every strategy employed by one player, the other has at least one sequence of moves that defeats it. Therefore, the game is undetermined.

$\square$